INTERACTIVE VISUALIZATION OF SPHERICAL TRIANGLES

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ABSTRACT

This article presents a novel software tool for the interactive visualization of spherical triangles. These triangles serve as the geometric foundation for spherical mechanism design; both synthesis and analysis. Moreover, the conceptualization and visualization of triangles facilitate the mastery of the synthesis and analysis of spherical mechanisms. This interactive visualization tool has been developed to facilitate the teaching and learning of spherical mechanisms at the graduate and undergraduate levels. The trigonometry employed and the interactive MATLAB implementation are presented. It is hoped that the dissemination of this software tool will facilitate the learning and advancement of spherical mechanism design techniques.

Keywords: spherical triangle; spherical trigonometry; spherical kinematics.

VISUALIZATION INTERACTIVE DES TRIANGLES SPHÉRIQUE

RÉSUMÉ

Cet article présente un nouvel outil programme pour la visualisation interactive des triangles sphériques. Les triangles sont le base géométrique pour la conception de mécanismes sphériques ; à la fois synthèse et analyse. La conceptualisation et la visualisation des triangles facilite la maîtrise de la synthèse et de l'analyse des mécanismes sphériques. Cet outil de visualisation interactif a été développé pour faciliter l'enseignement et l'apprentissage des mécanismes sphériques. La trigonométrie sphérique employée et l'implémentation interactive de MATLAB sont présentées. On espère que la diffusion de cet programme facilitera l'apprentissage et l'avancement des techniques de conception de mécanismes sphériques.

Mots-clés : triangles sphériques ; trigonométrie sphérique ; cinématique sphérique.

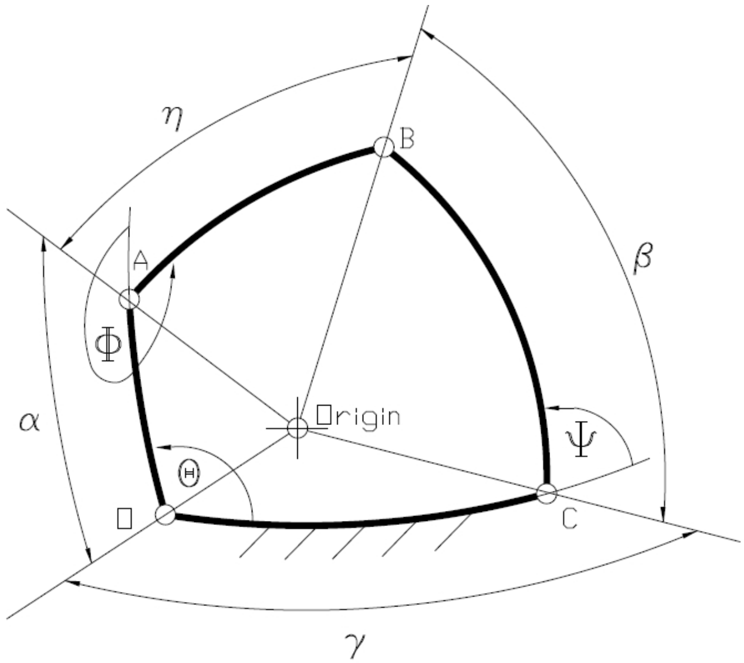

Fig. 1. A spherical four-bar mechanism.

1. INTRODUCTION

Spherical triangles are often encountered in the studies of spherical kinematics and in studies of the synthesis and analysis of spherical mechanisms. Most frequently one's first encounter with spherical triangles occurs in the study of the position analysis of spherical four-bar mechanisms, see Fig.1. Determining the interior angles and sides of spherical triangles *AOC* and *ABC* yields the complete position analysis of the spherical four-bar mechanism *OABC*. Additional examples of encountering spherical triangles include spherical pole triangles, spherical dyad triangles, and the like as shown in Figs. 2.

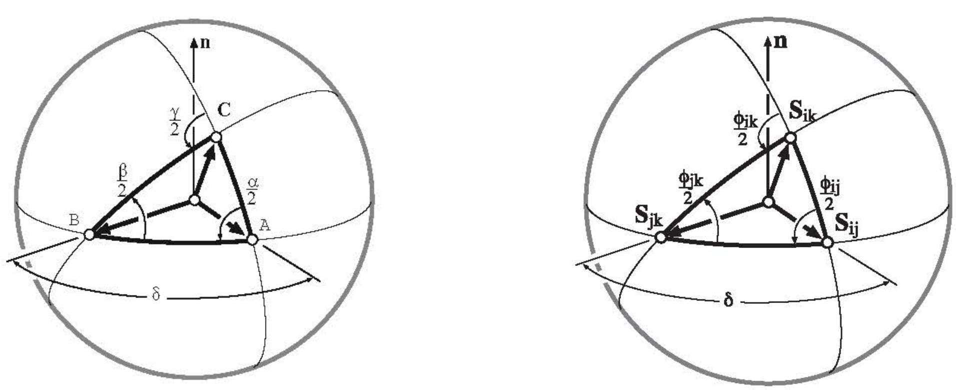

Fig. 2. Spherical triangles as illustrated in [1].

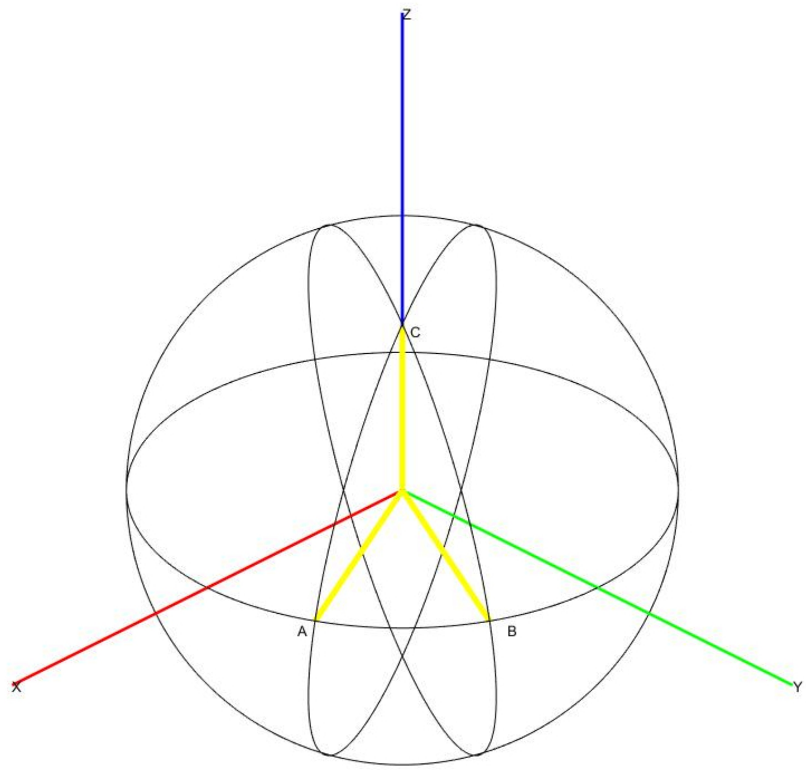

Fig. 3. A spherical triangle.

2. ANALYSIS OF SPHERICAL TRIANGLES

The study of spherical trigonometry dates back to ancient times, and its history is very well elucidated in [2]. Classical references on spherical trigonometry include [3–5]. The seminal work on spherical mechanisms is [6], and other works that discuss the analysis and synthesis of spherical mechanisms include [1, 7–12]. More recent works on the design of spherical mechanisms include [13–24].

A spherical triangle is shown in Fig. 3. Spherical triangles are defined by three great circles where each great circle is the intersection of the sphere with a plane passing through the center of the sphere. The three great circles appear in Fig. 3. There are six points on the surface of the sphere determined by the intersections of these great circles. These intersections define the vertices of a spherical traingle. In the figure, the nearest three intersections to the viewer are labeled A, B, and C. Their antipodal points being on the rear of the sphere are not labeled; however, they can be seen in Fig. 3. Let variables A, B, and C denote the interior angles at the three vertices of the spherical triangle ABC. The sides of the triangle have lengths a, b, and c where side a is opposite vertex A and is shown in the figure as a great arc from B to C. Similarly for sides b and c. The analysis of spherical triangles is accomplished by using the Law of Sines Eq. 1 and the Law of Cosines Eq. 2. Given any three sides or interior angles of a spherical triangle, these laws can be used to determine the three unknown sides or angles.

Law of Sines.

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C} \tag{1}$$

Law of Cosines.

$$\cos a = \cos b \cos c + \sin a \sin b \cos A \quad (2)$$

We define a fixed Cartesian coordinate frame with axes X, Y, and Z and origin at the center of the sphere as shown in Fig. 3.

2.1. Vertices: Cartesian Vectors

When Cartesian vectors are used to locate the vertices of a spherical triangle, denoted as $\vec{A}$, $\vec{B}$, and $\vec{C}$, confusion often arises in solving spherical triangles because the vertices do not appear in the Sine and Cosine Laws; only interior angles and sides do. In order to complete the analysis of such spherical triangles, the lengths of the three sides are computed from the Cartesian vectors that define the vertices per Eqs. 3.

$$\begin{aligned} a &= \arccos(\vec{B} \cdot \vec{C}) \\ b &= \arccos(\vec{A} \cdot \vec{C}) \\ c &= \arccos(\vec{A} \cdot \vec{B}) \end{aligned} \quad (3)$$

2.2. Vertices: Longitude & Latitude Angles

When the vectors that define the vertices of a spherical triangle, denoted as $\vec{A}$, $\vec{B}$, and $\vec{C}$, are located by their longitude and latitude angles, rotation matrices are used to compute their Cartesian coordinates per Eqs. 5. Once the Cartesian vectors have been determined, the lengths of the three sides of the triangle can be determined by using Eqs. 4.

$$\begin{aligned} \vec{A} &= [Rot_z(A_{lng})][Rot_y(-A_{lat})][1\ 0\ 0]^T \\ \vec{B} &= [Rot_z(B_{lng})][Rot_y(-B_{lat})][1\ 0\ 0]^T \\ \vec{C} &= [Rot_z(C_{lng})][Rot_y(-C_{lat})][1\ 0\ 0]^T \end{aligned} \quad (4)$$

2.3. Three Sides

When the lengths of the three sides of the spherical triangle are given, the three corresponding interior angles can be determined by using the Law of Cosines as detailed in Eqs. 5.

$$\begin{aligned} A &= \arccos\left(\frac{\cos(a) - \cos(b)\cos(c)}{\sin(b)\sin(c)}\right) \\ B &= \arccos\left(\frac{\cos(b) - \cos(c)\cos(a)}{\sin(c)\sin(a)}\right) \\ C &= \arccos\left(\frac{\cos(c) - \cos(a)\cos(b)}{\sin(a)\sin(b)}\right) \end{aligned} \quad (5)$$

2.4. Side-Angle-Side

When the lengths of two sides and their included angle of a spherical triangle are given, the two unknown interior angles and the one unknown side can be determined by using the Law of Cosines as detailed in Eqs. 6.

$$\begin{aligned} a &= \arccos(\cos(b)\cos(c) + \sin(b)\sin(c)\cos(A)) \\ B &= \arccos\left(\frac{\cos(b) - \cos(c)\cos(a)}{\sin(c)\sin(a)}\right) \\ C &= \arccos\left(\frac{\cos(c) - \cos(a)\cos(b)}{\sin(a)\sin(b)}\right) \end{aligned} \quad (6)$$

3. INTERACTIVE VISUALIZATION

In order to generate visualizations of spherical triangles, novel functions to draw arcs of great circles were created. The *generate arc points* function generates the Cartesian coordinates of points along the arc given the central axes that define the ends of the arc and the radius of the sphere.

```
function [point_x, point_y, point_z] = generate_arc_points(input_axis, ...
output_axis, radius, segments)
%Generate points along a spherical arc from input_axis to output_axis at
%specific radius, color, and # of segments
i = 1;
for t = 0:(1/segments):1.0,
    point = (t).*output_axis + (1-t).*input_axis;
    point = radius*(point/norm(point));
    point_x(i) = point(1);
    point_y(i) = point(2);
    point_z(i) = point(3);
    i = i + 1;
end
```

MATLAB's built in plot3 function is then used to draw the spherical arcs as shown below.

```
%   Draw the sides
[point_x, point_y, point_z] = generate_arc_points(A, B, 1.0, segments);
side_AB = plot3(point_x, point_y, point_z,'y-','linewidth',0.5*my_width);

[point_x, point_y, point_z] = generate_arc_points(B, C, 1.0, segments);
side_BC = plot3(point_x, point_y, point_z,'y-','linewidth',0.5*my_width);

[point_x, point_y, point_z] = generate_arc_points(A, C, 1.0, segments);
side_AC = plot3(point_x, point_y, point_z,'y-','linewidth',0.5*my_width);
```

3.1. User Interface

An interactive application for the visualization of spherical triangles has been developed in MATLAB [25]. Upon initialization, the user is presented a menu with four options for selecting the input data as shown in Fig. 4. Once the spherical triangle has been rendered, the user may interact with the three-dimensional graphics using MATLAB's standard tools such as *Rotate 3D*, *Pan*, *Zoom Out*, *Zoom In*, etc. Each of the three data input options are discussed below.

3.2. Vertices: Cartesian Vectors

Upon selection of the *3 Vertex Vectors* option, the user is presented with the graphical user interface seen in Fig. 5. Once the user has input their data and pushed the *Display Triangle* button, the triangle's interior angles and sides are presented per Fig. 6, and the interactive spherical triangle is presented in Fig. 7.

3.3. Vertices: Longitude & Latitude Angles

Selection of the *3 Vertex Angles* option results in the user being presented with the graphical user interface seen in Fig. 8. Once the user has input their data and pushed the *Display Triangle* button the triangle's interior angles and sides are presented per Fig. 9, and the interactive spherical triangle is presented in Fig. 10.

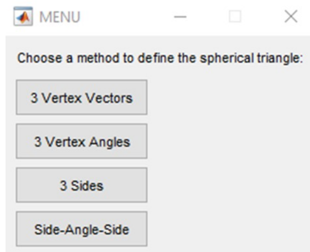

Fig. 4. Initial user interface for selection of the input data.

Fig. 5. User interface for defining 3 vertices using Cartesian vectors.

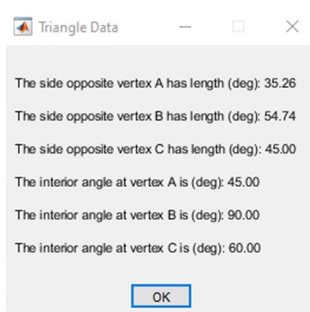

Fig. 6. Output data when defining 3 vertices using Cartesian vectors.

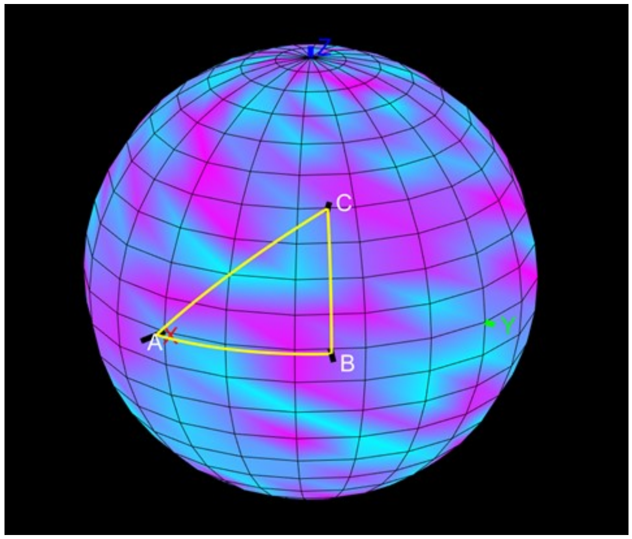

Fig. 7. Interactive spherical triangle when defining 3 vertices using Cartesian vectors.

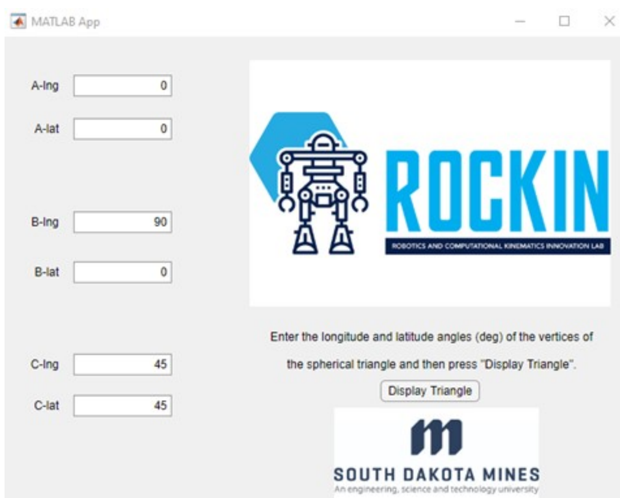

Fig. 8. User interface for defining three vertices using longitude and latitude angles.

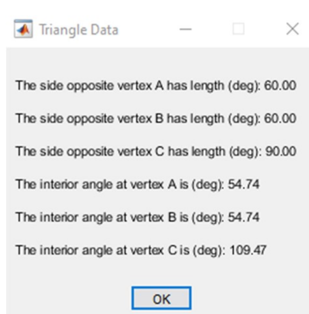

Fig. 9. Output data when defining three vertices using longitude and latitude angles.

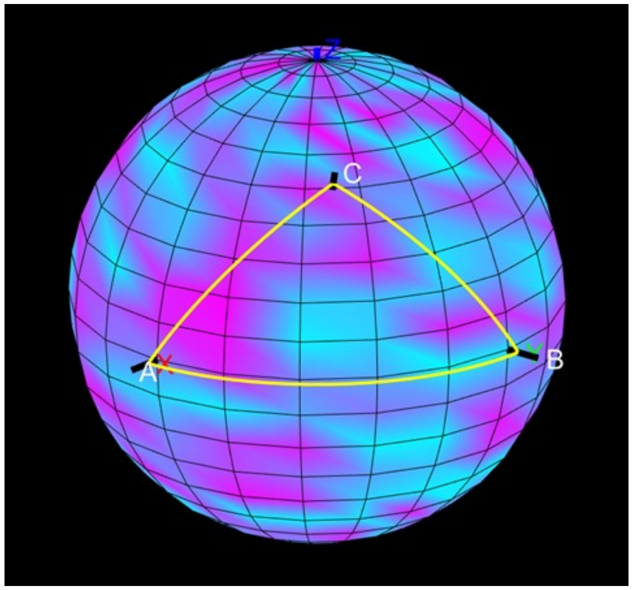

Fig. 10. Interactive spherical triangle when defining three vertices using longitude and latitude angles.

3.4. Three Sides

Selection of the *3 Sides* option results in the user being presented with the graphical user interface seen in Fig. 11. Once the user has input their data and pushed the *Display Triangle* button, the triangle's interior angles and sides are presented per Fig. 12, and the interactive spherical triangle is presented in Fig. 13. Note that $\vec{A}$ is arbitrarily located along the $x-axis$ and that $\vec{C}$ is located on the equatorial plane.

3.5. Side-Angle-Side

Selection of the *Side-Angle-Side* option results in the user being presented with the graphical user interface seen in Fig. 14. Once the user has input their data and pushed the *Display Triangle* button, the triangle's interior angles and sides are presented per Fig. 15, and the interactive spherical triangle is presented in Fig. 16. Note that $\vec{A}$ is arbitrarily located along the $x-axis$ and that $\vec{B}$ is located on the equatorial plane.

4. EXAMPLE

Let us revisit the position analysis of a spherical four-bar mechanism with nomenclature as shown in Fig.1. Consider Θ to be the input angle for a spherical four-bar mechanism with known link lengths $\alpha = 45$ (deg), $\eta = 55$ (deg), $\beta = 80$ (deg), and $\gamma = 70$ (deg). The kinematic position analysis entails finding the relative coupler angle Φ and the output angle Ψ as a function of the input angle $\Theta = 110$ (deg). This can be accomplished by analyzing the spherical triangle *AOC* in which two sides (α and γ) and their included angle Θ are known. Solving this triangle yields the side o as well as the interior angles at A and C. Next, the spherical triangle *ABC* is solved. In this triangle, all three sides are known (η, β, and o). Solving this triangle yields the interior angle at C, which in turn yields $\Psi = 83.79$ (deg), and the interior angle at A, which in turn yields the relative coupler angle $\Phi = 320.38$ (deg).

5. CONCLUSIONS

This article presented a novel software tool for the interactive visualization of spherical triangles. Spherical triangles serve as the geometric foundation for spherical mechanism design, both synthesis and analysis. The conceptualization and visualization of spherical triangles facilitate the mastery of the synthesis and anal-

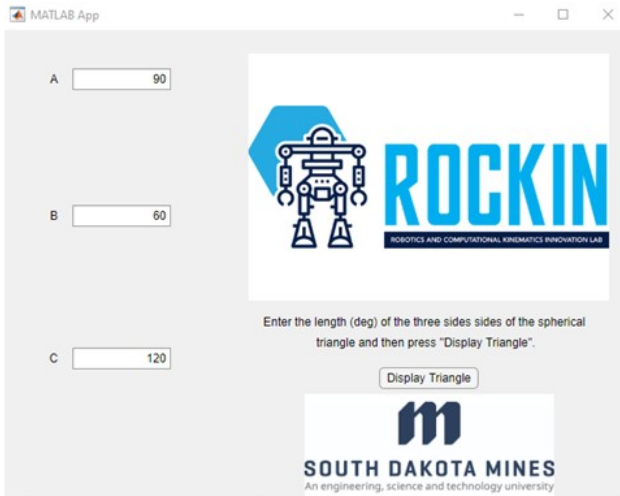

Fig. 11. User interface for defining three sides.

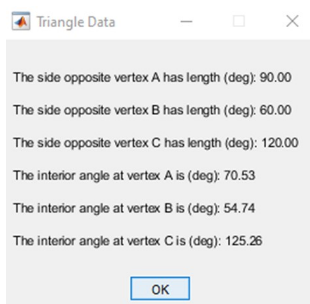

Fig. 12. Output data when defining three sides.

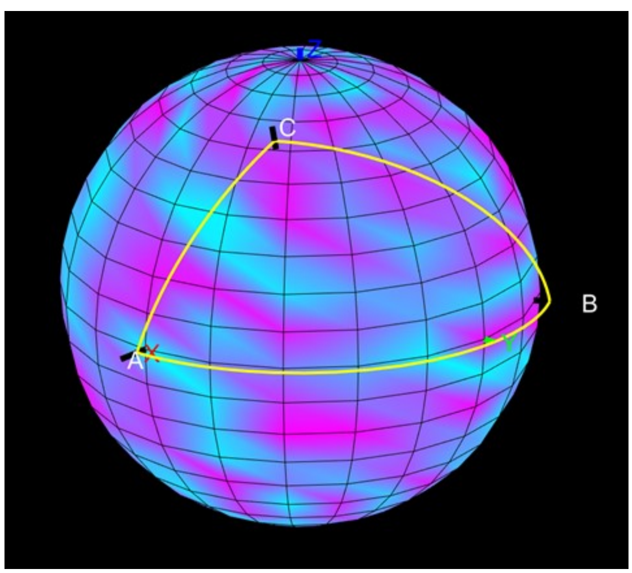

Fig. 13. Interactive spherical triangle when defining three sides.

Fig. 14. User interface for defining two sides and their included angle.

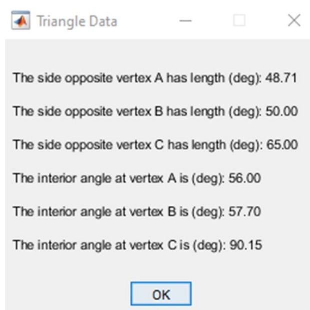

Fig. 15. Output data when defining two sides and their included angle.

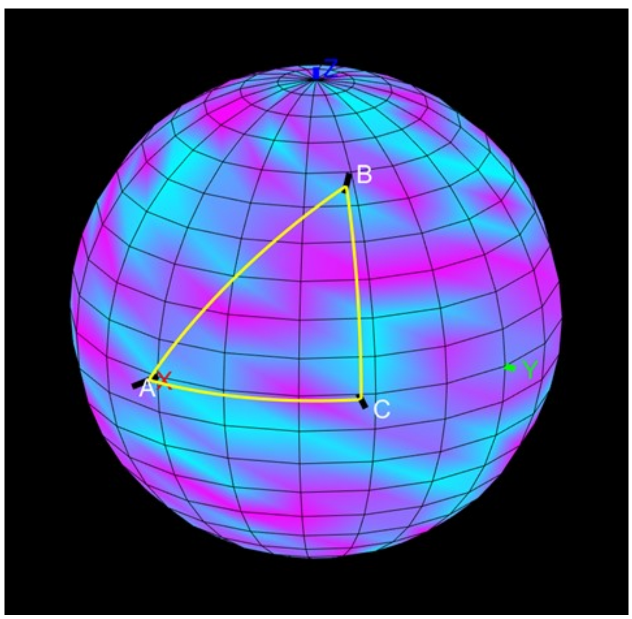

Fig. 16. Interactive spherical triangle when defining two sides and their included angle.

ysis of spherical mechanisms. This interactive visualization tool has been developed to facilitate the teaching and learning of spherical mechanisms at the graduate and undergraduate levels. The spherical trigonometry employed and the interactive MATLAB implementation were presented. It is hoped that the dissemination of this software tool will facilitate the learning and advancement of spherical mechanism design techniques. The MATLAB application presented here is freely available under the GNU General Public License [26] upon request to the author.

REFERENCES

1. McCarthy, J. and Soh, G. *Geometric Design of Linkages*. Interdisciplinary Applied Mathematics. Springer New York. ISBN 9781441978929, 2010.
   URL `https://books.google.com/books?id=jv9mQyjRIw4C`
2. Van Brummelen, G. *Heavenly Mathematics: The Forgotten Art of Spherical Trigonometry*. Princeton University Press. ISBN 9780691148922, 2013.
   URL `https://books.google.com/books?id=0BCCz8Sx5wkC`
3. Taylor, J. *Plane and Spherical Trigonometry*. Ginn, 1905.
   URL `https://books.google.com/books?id=yK8XAAAAIAAJ`
4. Palmer, C. and Leigh, C. *Plane and Spherical Trigonometry*. McGraw-Hill, 1916.
   URL `https://books.google.com/books?id=K9hKAAAAMAAJ`
5. Hann, J. *The Elements of Spherical Trigonometry*. J. Weale, 1849.
   URL `https://books.google.com/books?id=oMw2AAAAMAAJ`
6. Chiang, C. *Kinematics of Spherical Mechanisms*. Krieger Publishing Company. ISBN 9781575241555, 2000.
   URL `https://books.google.com/books?id=kVEOAAAACAAJ`
7. Gosselin, C.M. and Lavoie, E. "On the kinematic design of spherical three-degree-of- freedom parallel manipulators." *The International Journal of Robotics Research*, Vol. 12, No. 4, pp. 394–402. doi: 10.1177/027836499301200406, 1993.
   URL `https://doi.org/10.1177/027836499301200406`
8. Bai, S., Li, X. and Angeles, J. "A review of spherical motion generation using either spherical parallel manipulators or spherical motors." *Mechanism and Machine Theory*, Vol. 140, pp. 377–388. ISSN 0094-114X. doi:https://doi.org/10.1016/j.mechmachtheory.2019.06.012, 2019.
   URL `https://www.sciencedirect.com/science/article/pii/S0094114X19301223`
9. Hartenberg, R. and Denavit, J. *Kinematic Synthesis of Linkages*. McGraw-Hill series in mechanical engineering. McGraw-Hill. ISBN 9780070269101, 1964.
   URL `https://books.google.com/books?id=rfFSAAAAMAAJ`
10. Suh, C. and Radcliffe, C. *Kinematics and Mechanisms Design*. R.E. Krieger Publishing Company. ISBN 9780898746877, 1983.
   URL `https://books.google.com/books?id=Xe5SAAAAMAAJ`
11. Angeles, J. *Spatial Kinematic Chains: Analysis — Synthesis — Optimization*. Springer Berlin Heidelberg. ISBN 9783540113980, 1982.
   URL `https://books.google.com/books?id=OmdLAQAAIAAJ`
12. Ozgoren, M. *Kinematics of General Spatial Mechanical Systems*. Wiley. ISBN 9781119195733, 2020.
   URL `https://books.google.com/books?id=1T3NDwAAQBAJ`
13. Murray, A.P. and Larochelle, P.M. "A Classification Scheme for Planar 4R, Spherical 4R, and Spatial RCCC Linkages to Facilitate Computer Animation." Vol. Volume 1B: 25th Biennial Mechanisms Conference of *International Design Engineering Technical Conferences and Computers and Information in Engineering Conference*. doi:10.1115/DETC98/MECH-5887. V01BT01A079, 09 1998.
   URL `https://doi.org/10.1115/DETC98/MECH-5887`
14. Tse, D.M. and Larochelle, P.M. "A New Method of Task Specification for Spherical Mechanism Design." Vol. Volume 1B: 25th Biennial Mechanisms Conference of *International Design Engineering Technical Conferences and Computers and Information in Engineering Conference*. doi:10.1115/DETC98/MECH-5888. V01BT01A011, 09 1998.
   URL `https://doi.org/10.1115/DETC98/MECH-5888`

15. Larochelle, P.M. “SPADES: Software for Synthesizing Spatial 4C Mechanisms.” Vol. Volume 1B: 25th Biennial Mechanisms Conference of *International Design Engineering Technical Conferences and Computers and Information in Engineering Conference*. doi:10.1115/DETC98/MECH-5889. V01BT01A083, 09 1998.
URL `https://doi.org/10.1115/DETC98/MECH-5889`
16. Larochelle, P.M. “Synthesis of Planar RR Dyads by Constraint Manifold Projection.” Vol. Volume 2A: 24th Biennial Mechanisms Conference of *International Design Engineering Technical Conferences and Computers and Information in Engineering Conference*. doi:10.1115/96-DETC/MECH-1187. V02AT02A009, 08 1996.
URL `https://doi.org/10.1115/96-DETC/MECH-1187`
17. Ge, Q.J. and Larochelle, P.M. “Algebraic Motion Approximation With NURBS Motions and its Application to Spherical Mechanism Synthesis.” Vol. Volume 1B: 25th Biennial Mechanisms Conference of *International Design Engineering Technical Conferences and Computers and Information in Engineering Conference*. doi: 10.1115/DETC98/MECH-5881. V01BT01A009, 09 1998.
URL `https://doi.org/10.1115/DETC98/MECH-5881`
18. Ketchel, J.S. and Larochelle, P.M. “SphinxCAM: Computer-Aided Manufacturing for Spherical Mechanisms.” Vol. Volume 1B: 25th Biennial Mechanisms Conference of *International Design Engineering Technical Conferences and Computers and Information in Engineering Conference*. doi:10.1115/DETC98/MECH-5886. V01BT01A010, 09 1998.
URL `https://doi.org/10.1115/DETC98/MECH-5886`
19. Furlong, T.J., Vance, J.M. and Larochelle, P.M. “Spherical Mechanism Synthesis in Virtual Reality.” Vol. Volume 2: 24th Design Automation Conference of *International Design Engineering Technical Conferences and Computers and Information in Engineering Conference*. doi:10.1115/DETC98/DAC-5584. V002T02A037, 09 1998.
URL `https://doi.org/10.1115/DETC98/DAC-5584`
20. Tipparthi, H. and Larochelle, P. “A Hoop Method for Orientation Order Analysis of Spherical Mechanisms.” Vol. Volume 2: 34th Annual Mechanisms and Robotics Conference, Parts A and B of *International Design Engineering Technical Conferences and Computers and Information in Engineering Conference*, pp. 733–741. doi:10.1115/DETC2010-28072, 08 2010.
URL `https://doi.org/10.1115/DETC2010-28072`
21. Venkataramanujam, V. and Larochelle, P. “Approximate Motion Synthesis of Spherical Kinematic Chains.” Vol. Volume 8: 31st Mechanisms and Robotics Conference, Parts A and B of *International Design Engineering Technical Conferences and Computers and Information in Engineering Conference*, pp. 389–397. doi: 10.1115/DETC2007-34372, 09 2007.
URL `https://doi.org/10.1115/DETC2007-34372`
22. Larochelle, P.M., Schuler, J.M. and Ketchel, J.S. “SphinxCAM-Pro|E: Computer-Aided Modeling and Manufacturing of Spherical Mechanisms Via the Web.” Vol. Volume 2: 30th Annual Mechanisms and Robotics Conference, Parts A and B of *International Design Engineering Technical Conferences and Computers and Information in Engineering Conference*, pp. 339–347. doi:10.1115/DETC2006-99103, 09 2006.
URL `https://doi.org/10.1115/DETC2006-99103`
23. Murray, A.P. and McCarthy, J.M. “A Linkage Type Map for Spherical 4 Position Synthesis.” Vol. Volume 1: 21st Design Automation Conference of *International Design Engineering Technical Conferences and Computers and Information in Engineering Conference*, pp. 833–838. doi:10.1115/DETC1995-0109, 09 1995.
URL `https://doi.org/10.1115/DETC1995-0109`
24. Osborn, S.W. and Vance, J.M. “A Virtual Reality Environment for Synthesizing Spherical Four-Bar Mechanisms.” Vol. Volume 2: 7th International Conference on Design Theory and Methodology of *International Design Engineering Technical Conferences and Computers and Information in Engineering Conference*, pp. 885–892. doi:10.1115/DETC1995-0228, 09 1995.
URL `https://doi.org/10.1115/DETC1995-0228`
25. MATLAB. *version 9.9.0.1467703 (R2020b)*. The MathWorks Inc., Natick, Massachusetts, 2021.
26. GNU. “Gnu general public license.” `http://www.gnu.org/licenses/gpl.html`.